

Framed link presentations of 3-manifolds by an $O(n^2)$ algorithm, I: gems and their duals *

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Abstract

Given an special type of triangulation T for an oriented closed 3-manifold M^3 we produce a framed link in S^3 which induces the same M^3 by an algorithm of complexity $O(n^2)$ where n is the number of tetrahedra in T . The special class is formed by the duals of the *solvable gems*. These are in practice computationally easy to obtain from any triangulation for M^3 . The conjecture that each closed oriented 3-manifold is induced by a solvable gem has been verified in an exhaustible way for manifolds induced by gems with few vertices. Our algorithm produces framed link presentations for well known 3-manifolds which hitherto did not one explicitly known. A consequence of this work is that the 3-manifold invariants which are presently only computed from surgery presentations (like the Witten-Reshetkhin-Turaev invariant) become computable also from triangulations. This seems to be a new and useful result. Our exposition is partitioned into 3 articles. This first article provides our motivation, some history on presentation of 3-manifolds and recall facts about gems which we need.

1 Introduction

There are two main lines of presentations for 3-manifolds: the ones based on triangulations and the ones based on framed links surgery. These two types of presentations are complementary and so far, as long as we know, it is unknown how to go from a presentation of the first type to a presentation of the second by an efficient, effectively implementable algorithm. The goal of this work is to present such an algorithm. Its consequence is that triangulations and framed links presentations of 3-manifolds can finally be considered *computationally equivalent*. Costantino and Dylan [CT08] provide an $O(n^2)$ algorithm for a related problem: show that 3-manifolds efficiently bound 4-manifolds. Their algorithm has a large constant in its complexity bound and is not amenable for implementation. Our algorithm can be effectively implemented, does not mention 4-manifolds and has also complexity $O(n^2)$, where n is the number of tetrahedra in the input triangulation. Here no attempt is made to display a specific constant for the worse case performance. However, it becomes clear that this constant is small. The goal of this work is to prove the following theorem with an specific bound on the output of the algorithm.

(1.1) Theorem. *Given a resolvable gem with $2n$ vertices inducing a closed orientable 3-manifold M^3 , there is an $O(n^2)$ -algorithm producing a framed PL-link in \mathbb{R}^3 with at most n components which induces M^3 . Moreover, the number of 1-simplices that form the link is no more than $12n^2$. (The proof of this Theorem appears as Algorithm 1.7 in the third article.)*

The motivation for this work dates back to April/1993 where at a meeting in the Geometry Center in Minneapolis, Jeffrey Weeks (a former student of W. Thurston and the creator of SnapPea) asked the first author whether he had a framed link presentation of the Weber-Seifert hyperbolic dodecahedral space. He had not and apparently nobody else knew how to go, by an efficient computational scheme, from a explicit triangulation or Heegaard diagram of a 3-manifold to a surgery presentation of the same manifold. The present work produces such an scheme. The complexity $O(n^2)$ is just an upper bound for the worst case scenarium. In real cases the algorithm performs much better. Very recently, the Weber-Seifert dodecahedral hyperbolic space has been given much attention. It was proved to be non-Haken, [BRT12].

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In the fifth of a series of 8 papers in the beginning of the fifties Moise proved the fundamental fact that every 3-manifold admits a triangulation, [Moi52]. In 1962 Lickorish [Lic62] motivated by a question of Bing proved a result which has been basic in the presentation and investigation of closed oriented 3-manifolds since then. These two papers provide two complementary forms to present 3-manifolds, in a way to be made clear in the sequel. Even though Lickorish used Moise's theorem to obtain his in a constructive way, the approach (based on Heegaard decompositions and diagrams) is too topological to provide a usable combinatorial effectively implementable algorithm. A clear presentation of Lickorish's theorem appears in [Sti93].

The 3-manifolds treated here are closed and oriented, so Lickorish's result applies. This result was complemented 16 years later by the discovery of Kirby, which provided a set of two (non-local) moves which connects any two framed links inducing the same 3-manifold, [Kir78]. Kirby's calculus was reformulated in various useful ways, like the Fenn and Rourke approach, [FR79] or Kauffman's blackboard framed links, [Kau91]. Recently, in [Mar11], Martelli's provided a sufficient set of 4 local moves on framed links. This was a very surprising result, opening the way for the discovery of new invariants for 3-manifolds. All this research activity shows that 3-manifold theory can be seen as a subtle chapter in the theory of links.

The triangulation-based approach of 3-gems (introduced in [LM85]) provides a successful approach concerning explicitly census of 3-manifold: we are capable of finding canonical gem-representatives (named *attractors*) for 3-manifolds given by gems with few vertices. The approach is based on lexicography and worked successfully for gems with less than 30 vertices, without missings nor duplicates. The approach of ([Lin95]) can in principle be extended for gems with more vertices.

An algorithm to provide a gem directly from the drawing of a blackboard framed link inducing the same 3-manifold first appears in Chapter 13 of the joint monography of L. Kauffman and the first author, [KL94]. Each crossing in the link corresponds to 12 vertices in the gem. This algorithm is improved in L. Lins' thesis, [Lin07a], where the 12 vertices are reduced to 8. With this work the census of 3-manifolds given in [Lin95] based on gems and in [Lin07a] in terms of links finally can be compared back and forth. These two languages are complementary in the following sense: the gem approach, by its rich simplification theory based on lexicography and a combinatorial simplifying dynamics, is adequate in finding the homeomorphism between two 3-manifolds, enabling a proof that they are the same. The link approach permits us to compute the Witten-Reshetkhin-Turaev quantum invariants which are strong and frequently provide a proof that they are distinct. Together these approaches have been successful in providing census of 'small' 3-manifolds. Only two pairs of 3-manifolds remain unsolvable in the domain of L. Lins' thesis, [Lin07a].

As every 3-manifold seems to be induced by resolvable gems (the *Main Conjecture* of gem theory), our result, yielding an effective procedure to link the two languages is important in face of the connection of efficient algorithms and 3-manifolds, [Thu10].

The authors want to thank D. Thurston for having called their attention to reference [CT08] and for answering various questions in the context of this work. This paper completely reformulates previous Lins' work in the same topic, [Lin07b], making it obsolete. We are indebted to the Centro de Informática, UFPE/Recife, Brazil and to the Departamento de Matemática, UFPE/Caruaru, Brazil for financial support. Lins is also supported by a research grant from CNPq/Brazil, Proc. 301233/2009-8.

2 Gems and their duals

For background material on PL- and algebraic topology we refer to [Hat02] and [RS82]. Here we review the construction of a 3-manifold from a 3-gem and the basic facts that we need from this theory. More details in [LM85] and [Lin95].

2.1 Gems

A $(3+1)$ -graph G is a connected regular graph of degree 4 where to each vertex there are four incident differently colored edges in the color set $\{0, 1, 2, 3\}$. For $I \subseteq \{0, 1, 2, 3\}$, an I -residue is a component of the subgraph induced by the I -colored edges. Denote by $v(G)$ the number of 0-residues (vertices) of G . For $0 \leq i < j \leq 3$, an $\{i, j\}$ -residue is also called an ij -gon (it is an even polygon, where the edges are alternatively colored i and j). Denote by $b(G)$ the total number of ij -gons for $0 \leq i < j \leq 3$. Denote by $t(G)$ the total number of $\widehat{\{i\}}$ -residues for $0 \leq i \leq 3$, where the upper bar means complement on $\{0, 1, 2, 3\}$. A 3-gem is a $(3+1)$ -graph G satisfying $v(G) + t(G) = b(G)$. This relation is equivalent to having the vertices, edges and bigons restricted to any $\{i, j, k\}$ -residue inducing a plane graph where the faces are bounded by the bigons. Therefore we can embed

each such $\{i, j, k\}$ -residue into an sphere S^2 . We consider the ball bounded this S^2 as induced by the $\{i, j, k\}$ -residue. For this reason an $\{i, j, k\}$ -residue in a 3-gem, $i < j < k$, is also called a *triball*. An ij -gon appears once in the boundary of triball $\{i, j, k\}$ and once in the boundary of triball $\{i, j, h\}$. By pasting the triballs along disks bounded by all the pairs of ij -gons, $\{i, j\} \subset \{0, 1, 2, 3\}$ of a gem G , we obtain a closed 3-manifold denoted by $|G|$. The manifold is orientable if and only if G is bipartite, [LM85].

An $\{i, j, k\}$ -residue with 2 vertices is called a *blob over an h -colored edge*, or an h -edge, where $\{0, 1, 2, 3\} = \{h, i, j, k\}$. Suppose that some I -residue R has precisely 2 vertices u and v . R is an I -dipole if u and v are in distinct \hat{I} -residues. The cancelation of an I -dipole is the operation that (topologically) delete its edges and vertices and merge the $2|\hat{I}|$ free ends by identifying pairs along edges of the same colors in \hat{I} . The *creation of an I -dipole* is the inverse operation. A basic result in the theory of gems is the following result:

(2.1) Proposition. *Two gems induce the same 3-manifolds if and only if they are linked by a finite number moves, where each move is either a dipole creation or else a dipole cancellation.*

Proof. Complete proofs of this result for dimension n are in [FG82], [LM06]. □

An $|I|$ -residue is an I -residue for some $I \in \{0, 1, 2, 3\}$. Thus, an h -blob is a 3-residue. It follows from the definitions that a blob over h is an $\widehat{\{h\}}$ -dipole, $h = 0, 1, 2, 3$. Creation/cancellation of an h -blob is a “local move” because by definition of blob, their two ends are in distinct $\{h\}$ -residues. A 3-dipole creation/cancellation is a local move. The same is not true for 2- and 1-dipole creations/cancellations. The information necessary to conclude that they are indeed dipoles is spread in the whole graph. A great amount of research has been devoted to finding local moves. See for instance [LC99].

2.2 Twistors and their duals, hinges

Since our 3-manifolds are oriented, the gems inducing them are bipartite ([Lin95]) and their vertices are of two classes given by their parity.

Twistors in gems have been introduced in [Lin97]. But, for completeness, we recall their basics properties. Let (i, j, k) be a permutation of $(1, 2, 3)$. An i -twistor in a bipartite gem is a pair of vertices $\{u, v\}$ of the same parity such that they are in the same $0i$ -gon, the same jk -gon and in distinct $0j$ -, ik -, $0k$ - and ij -gons. The dual of twistors are called *hinges* and are formed by a pair of tetrahedra with a pair of opposite edges identified. A hinge is embedded into the M^3 induced by the gem. When we remove a hinge from M^3 , the resulting 3-manifold acquires an open toroidal hole in its boundary. In the gem this corresponds to deleting the pair of vertices u, v , thus producing 8 pendant edges. These edges correspond to 8 triangles forming the toroidal boundary of the resulting 3-manifold. Note that for a sufficiently small $\epsilon > 0$, an ϵ -neighborhood of a hinge corresponds to an embedded solid torus in M^3 . Twistors and hinges are central objects in this work. We also use the notion of *antipoles*. An i -antipole in a bipartite gem is a pair of vertices $\{u, v\}$ of distinct parity such that they are in the same $0i$ -gon, the same jk -gon and in distinct $0j$ -, ik -, $0k$ - and ij -gons.

The configuration which corresponds to a i -twistor in the dual is named a i -hinge. In Fig. 1 we present the dual steps involved in a 13-twisting.

For $i \neq j$, $\{i, j\} \subset \{1, 2, 3\}$, the ji -twisting of a j -twistor is the operation which exchange the i and j neighbors of u and v , see Fig. 2. Verify that the resulting configuration in the right side of Fig. 2 is an i -twistor by analysing the external connections in both sides of that figure.

(2.2) Proposition. *The ji -twisting a j -twistor is an internal operation in the class of bipartite gems.*

Proof. See Proposition 2 of [Lin97]: □

In the dual the operation ji -twisting corresponds to a Dehn-Lickorish twist (or surgery) in which an embedded solid torus in the induced 3-manifold is removed and glued back in a different way. It is a very enticing fact supporting gem theory that the simple combinatorial move of ji -twisting defines in a mathematically precise way exactly how to glue back the other solid torus. The solid tori involved are ϵ -neighborhoods of the hinges duals to the j -twistor and to the i -twistor.

Let e, f be a pair of edges of color c in a bipartite gem. The c -flipping at $\{e, f\}$ is the operation that switch the even (or odd) c -neighbors of the ends of e and f . The resulting $(3 + 1)$ -colored graph is not usually a gem. However, special pairs of flips produce gems as the proposition below shows. A labelling of the vertices of a gem is 0 -consecutive if the n pairs of 0-neighbors have the labelling $(1, 2), (3, 4), \dots, (2n - 1, 2n)$. These labelling are

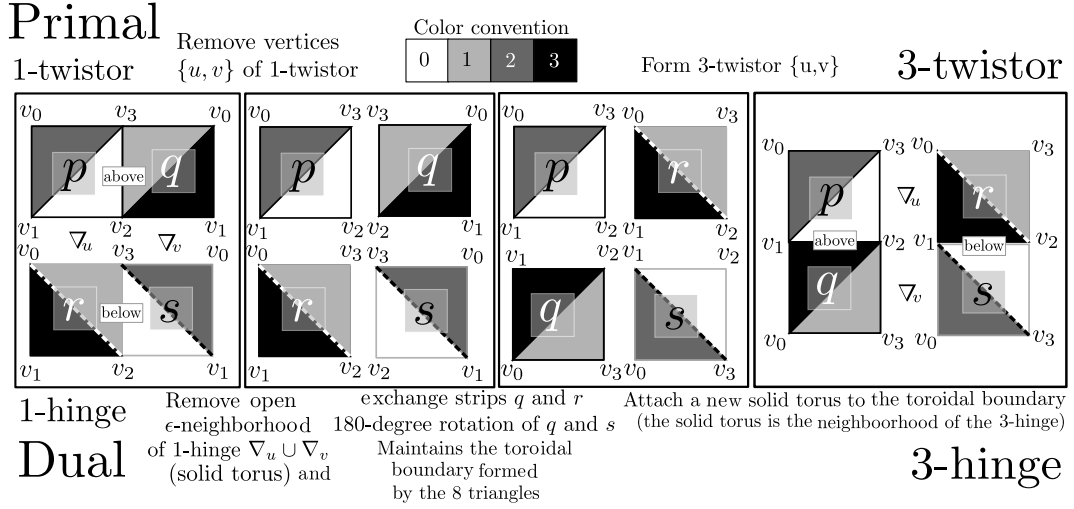


Figure 1: Tetrahedra ∇_u, ∇_v , hinges and strips: Steps involved in the dualization of a 13-twisting

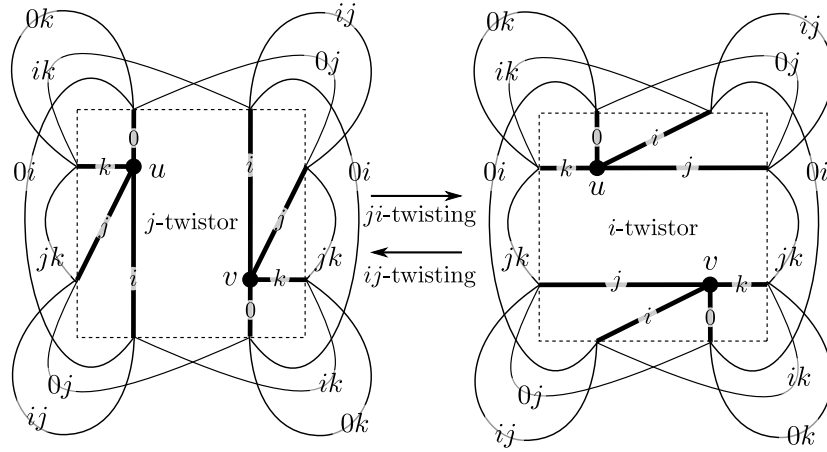


Figure 2: Obtaining an i -twistor by ji -twisting a j -twistor

important because with them the whole gem is defined by its $\hat{0}$ -residues. Blobs and special flips can play the role of dipoles, [LM06].

(2.3) Proposition. *Let G be a gem. (a) A ji -twisting of a j -twistor of G is factorable as one i -flip and one j -flip. (b) If G has a 0-consecutive labelling on its vertices, then a ji -twisting can be accomplished by one k -flip (which maintains planarity of the $\hat{0}$ -residue) followed by the $\{u, v\}$ label interchange. The final gem has a 0-consecutive labelling and so the ji -twisting is entirely depicted in the $\hat{0}$ -residue, a plane graph.*

Proof. We refer to Fig. 3. It is obvious that the passage from (A) to (E) is attainable by one i -flip and one j -flip, thus proving part (a). Note that A and B are the same configuration. The same is true for D and E. From B to C a k -flip is performed and C is no longer a gem nor it has a 0-consecutive labelling. This is easily fixed by the interchange of the labels u and v , which produces D and establishes (b). \square

A *crystallization* is a gem without 1-dipoles. Since ji -twisting does not disturb the number of 3-residues, the Proposition 2.2 holds with crystallization in the place of gem. Henceforth we work only with crystallizations. Denote by $b_{ij}(C)$ the number of ij -gons of C . The following proposition is implicitly used throughout this work.

(2.4) Proposition. *In any crystallization C the number of color complementary bigons are the same, i. e.,*

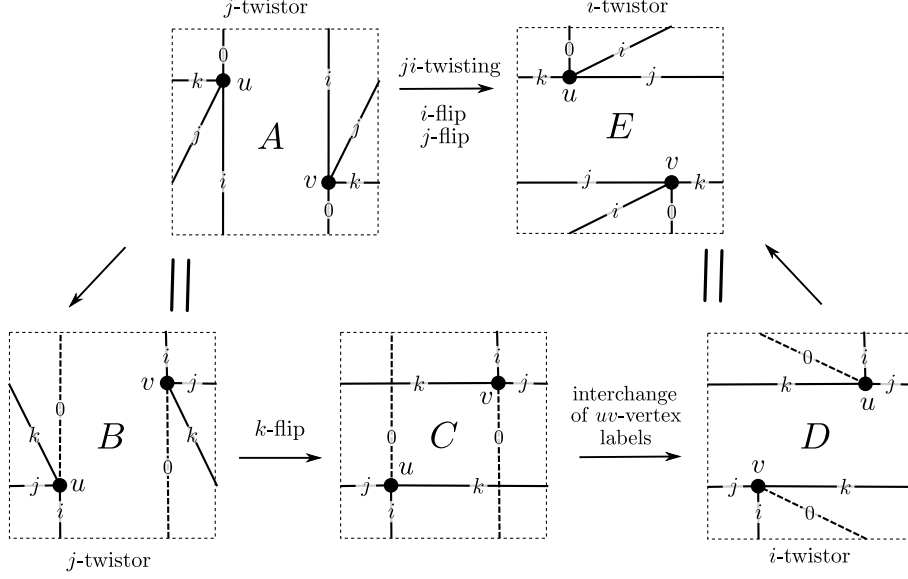


Figure 3: Factoring a ji -twisting by a k -flip and uv -label interchange (in the $\hat{0}$ -residue)

$$b_{01}(C) = b_{23}(C), b_{02}(C) = b_{13}(C) \text{ and } b_{03}(C) = b_{12}(C).$$

Proof. See [Lin86]. □

2.3 The gray graph of a crystallization and resolvable gems

For $i \in \{1, 2, 3\}$ the i -gray graph $T_i(C)$ of a crystallization C is defined as follows. The edge set of $T_i(C)$ are in 1-1 correspondence with the set of j - and k - twistors and antipoles of C . The vertices of $T_i(C)$ are in 1-1 correspondence with the the set of jk -gons of C . The incidence relation between vertices and edges is specified in the next Proposition, 2.5. It is convenient to present $T_i(C)$ as a *gray graph* over a planar (black) drawing of the \hat{i} -residue of C as in the example in the left side of Fig. 4, which is the 5th. *rigid gem* with 24 vertices inducing the 3-manifold EUCLID_1 , [Lin95]. In fact, we only depict a spanning tree of $T_i(C)$. Crystallization r_5^{24} with $(i, j, k) = (1, 2, 3)$ is used throughout the paper to illustrate all the ideas and constructions involved. Each vertex is inside the disk bounded by the bigon it corresponds to. Each edge is labelled by 3 integers in the format $tu-v$. Label $t \in \{j, k\}$ at edge e of $T_i(C)$ means that the corresponding twistor is a t -twistor; u and v are the labels of the vertices defining the twistor. See next proposition to conclude the definition of $T_i(C)$. By creating an adequate 2-dipole each antipole produces a twistor of the same type so that the corresponding edges in the gray graph have the same ends.

(2.5) Proposition. *The gray edge e of $T_i(C)$ corresponding to a j -twistor (respec. k -twistor) crosses two k -colored (respec. j -colored) edges of the two distinct jk -gons: the one incident to u and the one incident to v . The same is true with antipole in the place of twistor. Moreover, e has no other crossings with the \hat{i} -residue of C .*

Proof. Straightforward from the definitions. □

The two k -colored edges of C which are crossed by a gray edge e of $T_i(C)$ corresponding to a j -twistor form an e -pair.

(2.6) Corollary. *The manifestation in $T_i(C)$ of the ji -twisting of a j -twistor e with vertices u, v is simply the k -flip of the e -pair, followed by the interchange of labels u and v .*

Proof. Straightforward from Proposition 2.3. □

In general, $T_i(C)$ cannot be depicted free of crossings among its (gray) edges. A subset of n twistors is *disjoint* if the union of their pairs of vertices has cardinality $2n$. An i -resolution for a crystallization C is a

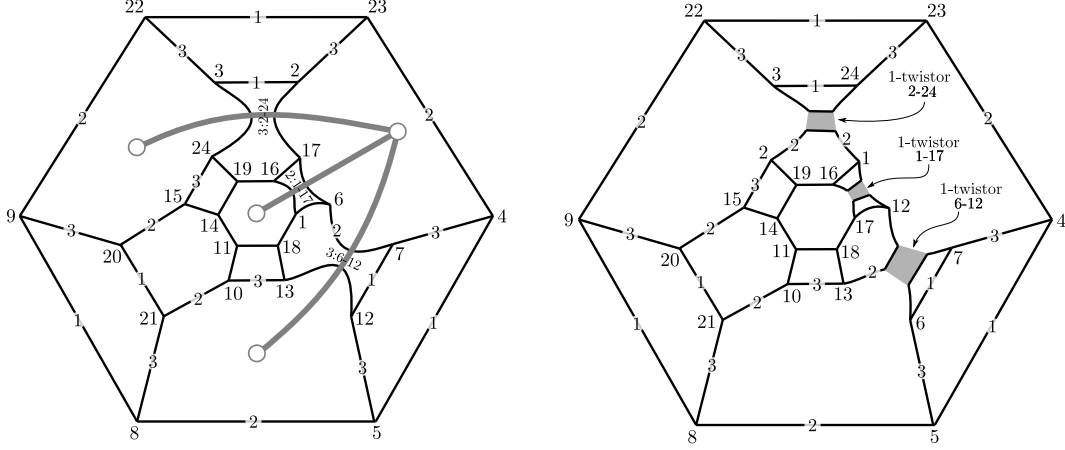


Figure 4: 1-resolution for r_5^{24} and its associated J^2 -gem $(r_5^{24})'$ (with a 0-consecutive labelling maintained)

disjoint subset of the j - and k -twistors inducing an spanning tree of $T(C)$ which is free of crossings. In the left side of Fig. 4 we depict a 1-resolution for r_5^{24} . A gem is *resolvable* if it is a crystallization and admits an i -resolution for some $i \in \{1, 2, 3\}$.

(2.7) Conjecture. *Any crystallization free of dipoles has an i -resolution for some $i \in \{1, 2, 3\}$.*

A proof of the above conjecture has been, so far, elusive. In view of the results presented in this paper we consider the above conjecture the most important open problem in gem theory. The resolution of the gem in Fig. 5 provides, with the theory to be here developed, a surgery presentation of the dodecahedral hyperbolic space, answering Jeffrey Weeks' specific question. In the picture, for $i \in \{2, 3\}$, a gray edge corresponds to an i -twistor if it crosses two $(5-i)$ -colored edges.

2.4 Pattern of intersection of two Jordan curves: J^2 -gems

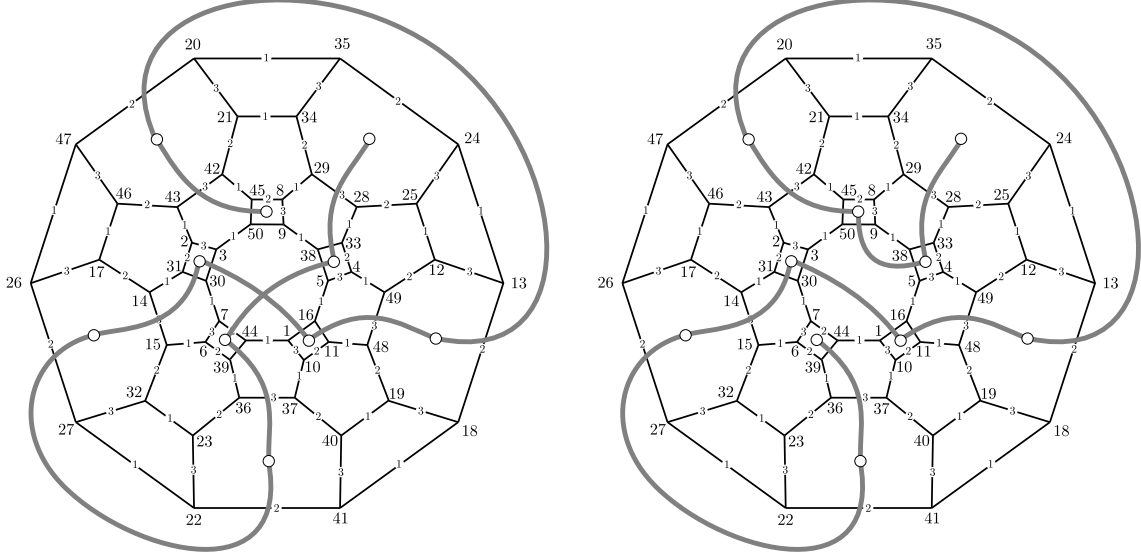
Let X and Y be two Jordan curves in the plane that intersect transversally at $2n$ points. We show that $X \cup Y$ defines naturally a gem inducing S^3 . Gems obtained in this way are crystallizations and induce S^3 . They are called J^2 -gems. This class of gems play a central role in our theory because we can PL-embed their duals in \mathbb{R}^3 by a polynomial algorithm. Moreover, because they are the final resulting gems of adequate twistings of the twistors in a resolution.

To color the segments induced by the crossings of X and Y paint the segments of X alternatively with colors j and k . Next, paint the segments of Y that go inside X with color 0 and the segments which go outside X with color i .

(2.8) Proposition. *The above coloring scheme yields a crystallization $C = X \cup Y$ inducing S^3 . Thus each J^2 -gem induces S^3 .*

Proof. Removing all the edges of a given color still yields a connected graph. So C has precisely four 3-residues. Denote by b_{ij} the number of ij -gons of C . Each one of these residues are planar graphs having $v = 2n$ vertices, $3v/2$ edges and $b_{12} + b_{13} + b_{23}$, $b_{02} + b_{03} + b_{23}$, $b_{01} + b_{13} + b_{03}$ and $b_{12} + b_{01} + b_{02}$ faces for, respectively, the $\widehat{0}$ -, $\widehat{1}$ -, $\widehat{2}$ -, $\widehat{3}$ -residue. Adding the four formulas for the Euler characteristic of the sphere imply that $v(C) + 4 = b(C)$. Therefore, C is a crystallization having one $0i$ -gon and one jk -gon. This implies that the fundamental group of the induced manifold is trivial: it is generated by $b_{0i} - 1 = 0$ generators, [Lin88]. Since Poincaré Conjecture is now proved, we are done. However, we can avoid using this fact and, as a bonus, obtaining the validity of next corollary, which is used in the sequel.

Assume that C is a J^2 -gem which does not induce S^3 and has the smallest possible number of vertices satisfying these assumptions. By planarity we must have a pair of edges C having the same ends $\{p, q\}$. Consider the graph $Cfus\{p, q\}$ obtained from C by removing the vertices p, q and the 2 edges linking them as well as welding the 2 pairs of pendant edges along edges of the same color. In [Lin85] we prove that if C is a gem, $C' = Cfus\{p, q\}$ is also a gem and that two exclusive relations hold regarding $|C|$ and $|C'|$, their induced



2-twistors: $\{11, 49\}, \{15, 31\}, \{29, 33\}, \{37, 39\}, \{43, 45\}$

3-twistors: $\{1, 3\}, \{1, 5\}, \{1, 7\}, \{1, 9\}, \{3, 5\}, \{3, 7\}, \{3, 9\}, \{5, 7\},$
 $\{5, 9\}, \{7, 9\}, \{18, 20\}, \{18, 22\}, \{18, 24\}, \{18, 26\}, \{20, 22\},$
 $\{20, 24\}, \{20, 26\}, \{22, 24\}, \{22, 26\}, \{24, 26\}$

Figure 5: This is a 50-vertex gem which behaves as the *attractor* (see [Lin95]) for the Weber-Seifert dodecahedral hyperbolic space. For a proof that it induces this space see also [Lin95]. On the left we do not have a resolution because of the crossing in the spanning tree. On the right we do have 1-resolution for the space: a crossing free spanning tree in the 1-gray graph of a gem inducing it. This makes the gem a resolvable one. We also list all of its 2- and 3-twistors. The gem is used to illustrate our algorithm yielding a link with 9 components (corresponding to the 9 edges of the spanning tree in the resolution). The output is a link having 68 1-simplices in its PL-embedding in \mathbb{R}^3 . We selected a projection with 142 crossings specified in Appendix B. The link is in a raw state and most likely can be substantially simplified by Reidemeister moves.

3-manifolds: either $|C| = |C'|$ in the case that $\{p, q\}$ induces a 2-dipole or else $|C| = |C'| \# (S^2 \times S^1)$. Since C' is a J^2 -gem, by our minimality hypothesis on C the valid alternative is the second. But this is a contradiction: the fundamental group of $|C|$ would not be trivial, because of the summand $S^2 \times S^1$. \square

The subgraph induced by vertices u and v of a gem G is denoted by $G[u, v]$. In general if $G[u, v]$ has two edges it might not be a 2-dipole: take the attractor for $S^2 \times S^1$ with 8 vertices, [Lin95]. However, we have

(2.9) Corollary. *If G induces S^3 and $G[u, v]$ has two edges, then it is a 2-dipole.*

Proof. Straightforward from the proof of the previous lemma: if it is not a 2-dipole the fundamental group of $|G|$ would not be trivial because of the summand $S^2 \times S^1$, contradicting [Lin88]. \square

Let T_i^j and T_i^k be the set of j -twistors and k -twistors in an i -resolution of a resolvable gem C . The drawing $T_i(C)$ over the 3-residues has the purpose of making evident the proof of the following essential Lemma:

(2.10) Lemma. *Let C be a resolvable gem with an i -resolution, T_i^j, T_i^k as above. Let C' be the graph obtained by ji -twisting in an arbitrary order an arbitrary subset of T_i^j and ki -twisting in arbitrary order an arbitrary subset of T_i^k . Then (a) C' is a gem and is independent of the order of the twistings. Moreover, (b) if we twist all members of $T_i^j \cup T_i^k$, then C' is a J^2 -gem.*

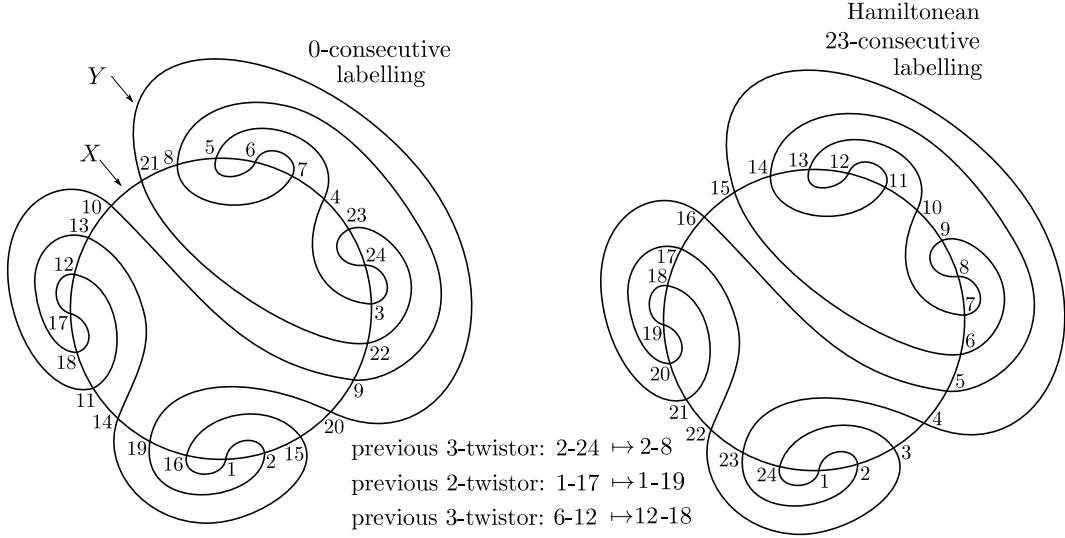


Figure 6: The J^2 -gem obtained from the 1-resolution of r_5^{24} and its hamiltonian 23-consecutive labelling

Proof. The essence of the proof can be followed in Fig. 3. The proof of (a) follows from Proposition 2.3 (enabling the twisting to become local plane moves – clearly independent of the order) and the proof of (b) from the fact that each twisting decreases by 1 the number of jk -gons. A gem with one jk -gon induces S^3 : cancelling its 3-dipoles we are left with a crystallization with one jk -gon, whence one $0i$ -gon (by Prop. 2.4). So, it is a J^2 -gem. \square

2.5 From a J^2 -gem \mathcal{J}^2 to an n -bloboid \mathcal{B} : sequence $\mathcal{H}_n, \mathcal{H}_{n-1}, \dots, \mathcal{H}_1$

We keep our convention that $(i, j, k) = (1, 2, 3)$. Given a 2-dipole $G[u, v]$ using colors 2, 3 and the 0-colored edges (resp. the 1-colored edges) e and f incident to u and v , the *thickening* of $G[u, v]$ is the 0-flip (respec. i -flip) of edges e and f . This produces a 3-dipole $G'[u, v]$ using the colors $\{0, 2, 3\}$ (resp. $\{1, 2, 3\}$). The thickening $G[u, v]$ into $G'[u, v]$ is called a *thickening of a 2-dipole into a 3-dipole*. An n -bloboid consists in a cyclic arrangement of $\{0, 1, 2\}$ -residues with 2 vertices (each a blob over a 3-edge). A J^2B -gem is one which becomes a J^2 -gem after their 3-dipoles are cancelled.

(2.11) Proposition. *Starting with a J^2 -gem \mathcal{J}^2 with $2n$ vertices we can arrive to an n -bloboid \mathcal{B} by means of $n - 1$ operations which thickens a 2-dipole into a 3-dipole, producing a sequence of J^2B -gems,*

$$(\mathcal{J}^2 = \mathcal{H}_n, \mathcal{H}_{n-1}, \dots, \mathcal{H}_2, \mathcal{H}_1 = \mathcal{B}).$$

Proof. The proof is by induction. For $\ell = n$ we have $\mathcal{H}_n = \mathcal{J}^2$ and so it is a J^2B -gem, establishing the basis of the induction. Assume that \mathcal{H}_ℓ is a J^2B -gem. For $\ell > 1$, let \mathcal{H}'_ℓ denote \mathcal{H}_ℓ after cancelling the blobs. Since \mathcal{H}'_ℓ is a J^2 -gem by the Jordan curve theorem a 2-dipole is present in it. The same 2-dipole is also present in \mathcal{H}_ℓ . Therefore it can be thickened, defining J^2B -gem $\mathcal{H}_{\ell-1}$, which establishes the inductive step. \square

The above proof is illustrated in Fig. 7. Note that the thickenings are of two types: either of type 0 where the edges e and f are 0-edges in the interior of the 23-gon, or else of type 1 if they are 1-edges in its exterior. The above sequence is by no means unique because various 2-dipole choices present themselves along the way. However, it is important to have as few as possible 0-1 or 1-0 transitions, as we shall see. Each thickening corresponds in the dual of the gems to a *pillow-balloon move* defined in the next section. Hence the labellings $(pb)^*$ in Fig. 7.

References

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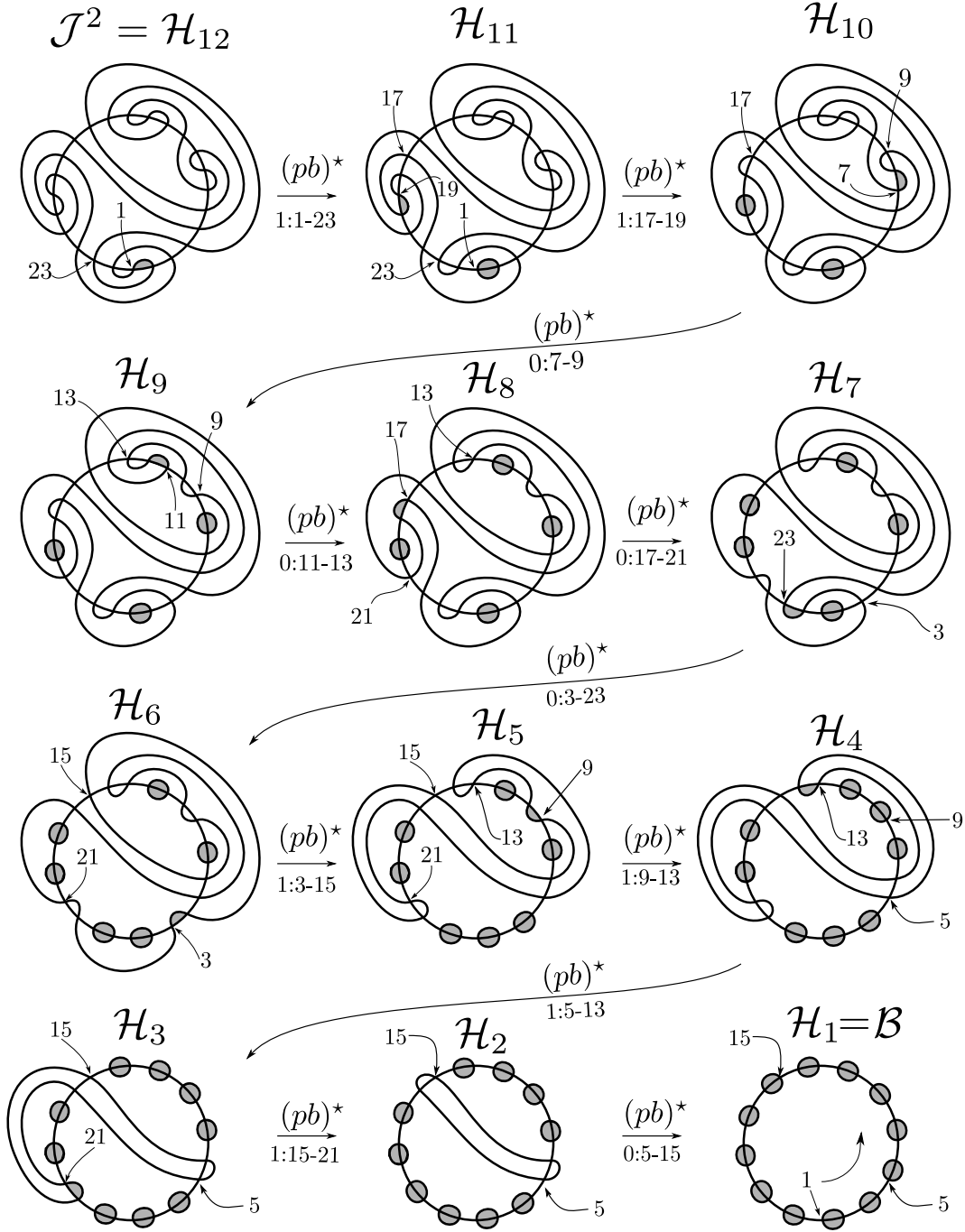


Figure 7: From the J^2 -gem \mathcal{J}^2 to the n -bloboid \mathcal{B} by a decreasing sequence of thickening 2-dipoles into 3-dipoles

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